SURFACES OF GENERAL TYPE AND $\mathfrak{sl}_2$-TRIPLES

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Abstract. The $\mathfrak{sl}_2$-triples play a fundamental role for the structure theory of Lie algebras, and representation theory in general. Here we investigate $\mathfrak{sl}_2$-triples of global vector fields on schemes X in positive characteristics $p > 0$, and develop a general theory for actions of the corresponding height-one group scheme $G = \mathrm{SL}_2[F]$. Sending a point to the Lie algebra of its stabilizer defines rational maps to various Grassmann varieties. For surfaces of general type, this yields fibrations in curves of genus $g \geq 2$ over the projective line. Using properties of the corresponding moduli stack $\mathscr{M}_g$, we prove that there are no smooth surfaces of general type with an $\mathfrak{sl}_2$-triple. On the other hand, employing Lefschetz pencils and Frobenius pullbacks we show that canonical surfaces of general type with such triples exist in abundance. In this connection, we classify the rational double points where the tangent sheaf is free or the evaluation pairing with Kähler differentials is surjetive, including characteristic two.

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Introduction

The main goal of this paper is to understand infinitesimal symmetries of surfaces of general type in positive characteristics. The general context is as follows:

Let X be a proper scheme over a ground field k, for the moment of arbitrary characteristic $p \geq 0$. Then $G = \mathrm{Aut}_{X/k}$ exists as a group scheme, the Lie algebra is the space $\mathfrak{g} = H^0(X, \Theta_{X/k})$ of global vector fields, and the connected component G^0 of the origin is of finite type. Brion and the first author [12] showed that every connected group scheme of finite type arises as some $\mathrm{Aut}^0_{X/k}$. On the other hand, for smooth varieties X of general type, $\mathrm{Aut}_{X/k}$ is actually finite, according to a result of

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Martin-Deschamps and Lewin-Ménégaux [32]. Thus $\mathrm{Aut}^0_{X/k}$ is a singleton, and the story in characteristic zero ends here. For $p>0$, however, this finite group scheme might be non-reduced, but not much seems to be known about its structure. In light of this state of affairs it is natural to concentrate attention on the Frobenius kernel $G[F]=\mathrm{Aut}^0_{X/k}[F]$, or equivalently on the restricted Lie algebra $\mathfrak{g}=H^0(X,\Theta_{X/k})$, and try to understand that.

From now on we are in characteristic $p>0$. The above problem is vacuous in dimension $n=1$, in other words for curves of genus at least two, because then the tangent sheaf has strictly negative degree. The situation changes drastically in dimension $n=2$: The first surfaces of general type with non-zero global vector fields where constructed by Raynaud [41]. Further results in this direction where obtained by Shepherd-Barron [45] and the second author [47], [48]. We showed in [44] that for smooth surfaces of general type, and their canonical models alike, among the startling multitude of restricted Lie algebras only k^n and $k^n\rtimes\mathfrak{gl}_1(k)$ and $\mathfrak{sl}_2(k)$ are possible; the corresponding height-one group schemes are the Frobenius kernels of the vector group $\mathbb{G}_a^n$, the semidirect product $\mathbb{G}_a^n\rtimes\mathbb{G}_m$, and the special linear group SL_2, respectively. For all of them we constructed example, with the nerve-wrecking exception of $\mathfrak{sl}_2(k)$.

The restricted Lie algebra $\mathfrak{sl}_2(k)$ is three-dimensional, and plays a fundamental role in the structure theory of Lie algebras and representation theory in general (confer [10], Chapter VIII, §1 and §11). However, applications in algebraic geometry are surprisingly scarce. The traceless matrices $e=\left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right)$ and $h=\left(\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right)$ and $f=\left(\begin{smallmatrix}0&0\\-1&0\end{smallmatrix}\right)$ form a basis, with bracket and p-map given by

$$[h,e]=2e,\quad [h,f]=-2f,\quad [e,f]=-h\quad\text{and}\quad e^{[p]}=f^{[p]}=0,\quad h^{[p]}=h.$$

For any restricted Lie algebra $\mathfrak{h}$, the injective homomorphisms $\mathfrak{sl}_2(k)\to\mathfrak{h}$ correspond to linearly independent triples (e,h,f) from $\mathfrak{h}$ satisfying the above relations. Following Dynkin and Bourbaki we call them *$\mathfrak{sl}_2$-triples.* We also find it convenient to say that a scheme X *admits an $\mathfrak{sl}_2$-triple* if the restricted Lie algebra $\mathfrak{h}=H^0(X,\Theta_{X/k})$ of global vector fields contains such an (e,h,f). Geometrically speaking, this is nothing but a faithful action of the height-one group scheme $\mathrm{SL}_2[F]$. Our first main result is:

Theorem A. *(See Thm. 4.1) No smooth surface X of general type admits an $\mathfrak{sl}_2$-triple.*

To establish this, we develop a general theory for actions of the height-one group scheme $G=\mathrm{SL}_2[F]$ on integral schemes Y whose generic stabilizer has order $|G_\eta|=p^2$. Writing $\mathfrak{g}=\mathfrak{sl}_2(k)$, we then consider the so-called *inertia map*

$$f_{\mathrm{inert}}:Y\dashrightarrow\mathrm{Grass}^2(\mathfrak{g}),\quad y\longmapsto\mathrm{Lie}(G_y)$$

to the Grassmann variety of two-dimensional subspaces. This of course factors over the scheme of two-dimensional subalgebras, which turn out to be the curve $B\subset\mathrm{Grass}^2(\mathfrak{g})$ defined by a quadratic equation $T_0^2-4T_1T_2=0$. From this one can already deduces that the characteristic must be $p\neq2$. Now the key insight is:

Theorem B. *(See Thm. 3.5) On some G-modification $\tilde{Y}$, the action is fixed point free, and the foliation $\mathfrak{g}\cdot\mathscr{O}_{\tilde{Y}}\subset\Theta_{\tilde{Y}/k}$ is a direct summand identified with $\tilde{f}^*_{\mathrm{inert}}(\Theta_{B/k})$.*

For smooth surfaces X of general type this leads, at least on some modification, to a pencil of smooth curves of genus $g \geq 2$. By Szpiro's generalization [46] of results of Parshin [39] and Arakelov [2], the classifying morphism $\mathbb{P}^1 \to \mathscr{M}_g$ to the moduli stack must be constant. But this implies $X = C \times \mathbb{P}^1$, which is impossible for surfaces of general type.

However, the story does not end here: Recall that each *smooth surface of general type* S has a unique model Y where the dualizing sheaf ω_Y is ample and the singularities $\mathscr{O}_{Y,y}$ are rational double points. Let us call such Y *canonical surfaces of general type*. From a certain perspective they are of superior interest, because they form an Artin stack with finite inertia groups. It is also convenient to have a name for their modifications $X \to Y$ whose singularities $\mathscr{O}_{X,x}$ stay rational double points; we propose to call them *RDP surfaces of general type*. Note that Blanchard's Lemma gives an inclusion $\mathrm{Aut}^0_{X/k} \subset \mathrm{Aut}^0_{Y/k}$, which may or may not be an equality. Our third main results shows that such surfaces with $\mathfrak{sl}_2$-triples are surprisingly common:

Theorem C. *(See Thm. 5.1) Let S be a smooth surface of general type in characteristic $p \geq 3$. Then there is a purely inseparable alteration $X \to S$ by some RDP surface of general type having an $\mathfrak{sl}_2$-triple. Moreover, one may assume that the tangent sheaf $\Theta_{X/k}$ is locally free, and the action of $\mathrm{SL}_2[F]$ has no fixed points.*

The idea is to use *Lefschetz pencils* $C_t \subset S$, which are parameterized by $t \in \mathbb{P}^1$ and becomes a fibration upon blowing-up the axis Z; one then obtains the desired RDP surface of general type $X = \mathrm{Bl}_Z(S) \times_{\mathbb{P}^1} (\mathbb{P}^1, F)$ as a *relative Frobenius base-change of the Lefschetz fibration*. The $\mathrm{SL}_2[F]$-action arises from the canonical action on the second factor. Note that using higher Frobenius powers F^n produces actions of iterated Frobenius kernels. Also note that the fibration $X \to \mathbb{P}^1$ can be seen as morphism $\mathbb{P}^1 \to \bar{\mathscr{M}}_g$ into the Delinge–Mumford stack of stable curves which factors over the relative Frobenius of the relative Frobenius map of the projective line. We believe that the phenomenon of inseparability in classifying maps to moduli stacks deserve further study.

The above construction produces rational double points whose tangent modules $\Theta_{R/k}$ are free and the evaluation pairing $\Theta_{R/k} \otimes \Omega^1_{R/k} \to R$ is surjective. In light of Artin's classification of rational double points [5], this raises the question which rational double points have one or both of these properties. Such question go back to Lipman [28], and arise in many different contexts ([49], [20], [42], [24], [43], [21], [26], [27], [31]). Combining the theory of *minimal free resolutions* with techniques from *Gröbner bases*, we indeed settle this, including the most challenging case of characteristic two:

Theorem D. *(See Section 6) A rational double point in characteristic $p > 0$ has free tangent module $\Theta_{R/k}$ or surjective evaluation pairing $\Theta_{R/k} \otimes \Omega^1_{R/k} \to R$ if and only if occurs in table 1.*

Note that for RDP surfaces of general type with faithful $\mathfrak{sl}_2$-triple, only $p \geq 3$ matters. It would be interesting to see whether E_8^0 ($p = 3, 5$) and E_6^0, E_7^0 ($p = 3$) really occur in constructions like $X = \mathrm{Bl}_Z(S) \times_{\mathbb{P}^1} (\mathbb{P}^1, F)$, perhaps with certain pencils that violate the Lefschetz condition.

The paper is organized as follows: In Section 1 we discuss some relevant facts on automorphism group schemes, surfaces of general type, rational double points, and

RDP	condition	tangent module free	pairing surjective
A_l	$l \equiv -1 \bmod p$	yes	yes
D_{2n}^0, D_{2n+1}^0	$p=2$	yes	yes
$D_{2n}^1, \ldots, D_{2n}^{n-1}$		yes	no
$D_{2n+1}^1, \ldots, D_{2n+1}^{n-1}$		no	no
E_8^0	$p=2,3,5$	yes	yes
E_6^0, E_7^0	$p=2,3$	yes	yes
$E_6^1, E_7^1, E_8^1, E_8^2$	$p=2$	yes	no

TABLE 1. Properties of rational double points in positive characteristic

$\mathfrak{sl}_2(k)$ as restricted Lie algebra. In Section 2 we analyze fibrations $f: X \to \mathbb{P}^1$ that are equivariant with respect to actions of the height-one group scheme $G = \mathrm{SL}_2[F]$. Our core observations appear in Section 3, where we analyze the stabilizers for G-actions on schemes X, and the resulting rational maps to Grassmann varieties of subalgebras in $\mathfrak{g} = \mathfrak{sl}_2(k)$. These results are applied to smooth surfaces of general type in Section 4, where the above rational maps are interpreted in term of the moduli stack $\mathscr{M}_g$. Section 5 contains construction of canonical surfaces of general type having $\mathfrak{sl}_2$-triples. In Section 6 we use homological algebra and Gröbner bases techniques to determine which rational double points have free tangent module $\Theta_{R/k}$ or where the evaluation pairing $\Theta_{R/k} \otimes \Omega^1_{R/k} \to R$ is surjective. In the final Section 7 we classify the rational double points admitting an $\mathfrak{sl}_2$-triples and study when they are fixed point free.

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1. Generalities

Let k be a ground field, for the moment of arbitrary characteristic $p \geq 0$, and X be a proper scheme. Matsumura and Oort [35] established that $\mathrm{Aut}_{X/k}$ is representable by a group scheme whose connected component $\mathrm{Aut}^0_{X/k}$ of the origin is of finite type. The Lie algebra is the space $H^0(X, \Theta_{X/k})$ of global vector fields, where $\Theta_{X/k} = \underline{\mathrm{Hom}}(\Omega^1_{X/k}, \mathscr{O}_X)$ denotes the *tangent sheaf*. This is a coherent sheaf, which is locally free provided that X is smooth, and than rank equals dimension. Note that in positive characteristics there are also singular schemes having locally free tangent sheaves.

Let us say that a morphism $f : X \to Y$ between proper schemes is *in Stein factorization* if the canonical map $\mathscr{O}_Y \to f_*(\mathscr{O}_X)$ is an isomorphism. Then Blanchard's Lemma ([8], Proposition 1.1, see [11], Theorem 7.2.1 for the scheme-theoretic version) ensures that there is a unique homomorphism

$$f_* : \mathrm{Aut}^0_{X/k} \longrightarrow \mathrm{Aut}^0_{Y/k}$$

making f equivariant with respect to $G = \mathrm{Aut}^0_{X/k}$. The induced map of Lie algebras is likewise written as $f_* : H^0(X, \Theta_{X/k}) \to H^0(Y, \Theta_{Y/k})$. These maps are monomorphisms provided there is a G-stable schematically dense open set $U \subset X$ such that $f|U$ is an isomorphism to a schematically dense open set $V \subset Y$. We then regard the maps as inclusions $\mathrm{Aut}^0_{X/k} \subset \mathrm{Aut}^0_{Y/k}$ and $H^0(X, \Theta_{X/k}) \subset H^0(Y, \Theta_{Y/k})$.

Recall that a smooth proper scheme X is called *variety of general type* provided $h^0(\mathscr{O}_X) = 1$ and the dualizing sheaf ω_X is a *big invertible sheaf*. Roughly speaking, this means that the function $t \mapsto h^0(\omega_X^{\otimes t})$ has a growths rate like a monomial of degree $n = \dim(X)$. According to a result of Martin-Deschamps and Lewin-Ménégaux [32], the group scheme $\mathrm{Aut}_{X/k}$ is then finite.

The *smooth surfaces of general type* S can also be characterized by the condition that $(\omega_S \cdot D) > 0$ for every movable curve $D \subset S$. Each such surface comes with a *canonical model* Y, where ω_Y is an ample invertible sheaf and the singularities $\mathscr{O}_{Y,y}$ are *rational double points*. Let us call these Y *canonical surfaces of general type*. It is convenient to have a designation for the modifications $X \to Y$ whose singularities $\mathscr{O}_{X,x}$ are rational double points; we propose to call them *RDP surfaces of general type*. Special cases are the *minimal surface of general type*, which arise from S by successively contracting (-1)-curves. For more information, we refer to the monographs [7] and [6]. Let us record the following:

Proposition 1.1. *For each RDP surface X of general type, the group scheme $\mathrm{Aut}_{X/k}$ is finite.*

Proof. Write $G = \mathrm{Aut}_{X/k}$ and let $S \to X$ be the minimal resolution of singularities. If the ground field k is perfect, S is a smooth surface of general type, the subgroup scheme G_{red} extends to S, and it follows that G_{red} and hence G is finite. Over imperfect fields in characteristic $p > 0$ we argue as follows: Choose $n \geq 0$ so that $G' = G/G[F^n]$ is smooth. The latter acts on the corresponding iterated Frobenius pullback $X' = X^{(p^n)}$. The smooth group scheme G' extends to the normalization S' and minimal resolution S'' of X'. For n sufficiently high, the surfaces S' and S'' are geometrically normal and geometrically regular, respectively. This shows that G' and hence also G are finite. □

From now on we assume $p > 0$. For each group scheme G the Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$ has, besides the bracket $[x, y]$ defined as commutator in the ring of differential operators, also a *p-map* $x^{[p]}$ stemming from p-fold composition of differential operators, and thus becomes a *restricted Lie algebra* (see [17], Chapter II, §7 or [44], Section 1 for the axioms). A vector $x \in \mathfrak{g}$ is called *p-closed* if the line $\mathfrak{h} = kx$ is stable under the p-map, in other words $x^{[p]} = \lambda x$ for some scalar λ, and thus becomes a commutative restricted subalgebra. In case $\lambda \neq 0$ the vector is called *multiplicative*, otherwise *additive*. By the *Demazure–Gabriel Correspondence* ([17], Chapter II, §7, Theorem 3.5) the functor $G \mapsto \mathfrak{g}$ induces an equivalence between the categories of group schemes of finite type annihilated by the relative Frobenius and the finite-dimensional restricted Lie algebras. For convenience such G are called *group schemes of height one*. The group schemes $\alpha_p^n = \mathbb{G}_a^n[F]$ correspond to the standard vector space $\mathfrak{g} = k^n$, where both bracket and p-map are trivial, while $\mu_p = \mathbb{G}_m[p]$ corresponds to $\mathfrak{g} = \mathfrak{gl}_1(k)$, which is the one-dimensional standard vector space with p-map $\lambda^{[p]} = \lambda^p$.

Recall that each associative algebra A becomes a restricted Lie algebra, via $[x,y] = xy - yx$ and $x^{[p]} = x^p$. Write $\mathfrak{gl}_n(k)$ for the restricted Lie algebra arising form $A = \mathrm{Mat}_n(k)$. It sits in two short exact sequences with kinks

$$
\begin{array}{ccccccc}
0 \longrightarrow \mathfrak{gl}_1(k) & & \overset{n}{\dashrightarrow} & & \mathfrak{gl}_1(k) \longrightarrow 0 \\
& \searrow & & \nearrow & \\
& & \mathfrak{gl}_n(k) & & \\
& \nearrow & & \searrow & \\
0 \longrightarrow \mathfrak{sl}_n(k) & & \dashrightarrow & & \mathfrak{pgl}_n(k) \longrightarrow 0,
\end{array} \tag{1}
$$

where the inclusion of $\mathfrak{gl}_1(k)$ is given by the scalar matrices, and the surjection to $\mathfrak{gl}_1(k)$ is the trace map. In turn, we get an exact sequence

$$0 \longrightarrow \mathrm{Ker}(n|\,\mathfrak{gl}_1(k)) \longrightarrow \mathfrak{sl}_n(k) \longrightarrow \mathfrak{pgl}_n(k) \longrightarrow \mathrm{Coker}(n|\,\mathfrak{gl}_1(k)) \longrightarrow 0,$$

where the outer terms are either copies of $\mathfrak{gl}_1(k)$ or vanish, depending on the p-divisibility of n.

Throughout, we are particularly interested in the restricted Lie algebra $\mathfrak{sl}_2(k)$, where the matrices $h = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ and $e = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ and $f = \left(\begin{smallmatrix} 0 & 0 \\ -1 & 0 \end{smallmatrix}\right)$ form a basis. Bracket and p-map are determined by

$$[h,e] = 2e, \quad [h,f] = -2f, \quad [e,f] = -h \quad \text{and} \quad e^{[p]} = f^{[p]} = 0, \quad h^{[p]} = h. \tag{2}$$

For each traceless matrix $N = \left(\begin{smallmatrix} 0 & -d \\ 1 & 0 \end{smallmatrix}\right)$ in rational normal form one has $N^2 = -dE$, which for $p \geq 3$ implies $N^p = (-d)^{(p-1)/2}N$. Using that every traceless matrix is similar to such an N, we obtain

$$x^{[p]} = \begin{cases} (\alpha^2 - \beta\gamma)^{(p-1)/2} \cdot x & \text{if } p \neq 2; \\ (\alpha^2 - \beta\gamma) \cdot h & \text{else,} \end{cases} \tag{3}$$

for each vector $x = \alpha h + \beta e + \gamma f$ with coefficients $\alpha, \beta, \gamma \in k$. In turn, the homogeneous quadratic equation $\alpha^2 - \beta\gamma = 0$ defines the locus of additive vectors $x \in \mathfrak{sl}_2(k)$, at least set-theoretically. The following two results are well-known, and we leave the verification to the reader.

Proposition 1.2. *For $p \neq 2$, the canonical map $\mathfrak{sl}_2(k) \to \mathfrak{pgl}_2(k)$ is bijective, every one-dimensional subspace $\mathfrak{h} = kx$ is a restricted subalgebra, and none of them is an ideal. Moreover, there are no two-dimensional ideals.*

This is in marked contrast to the situation at the remaining prime:

Proposition 1.3. *For $p = 2$ the kernel of $\mathfrak{sl}_2(k) \to \mathfrak{pgl}_2(k)$ is generated by h, the image is a copy of k^2, and the resulting extension $0 \to \mathfrak{gl}_1(k) \to \mathfrak{sl}_2(k) \to k^2 \to 0$ does not split. A vector $x = \alpha h + \beta e + \gamma f$ generates a restricted subalgebra if and only if $\alpha^2 - \beta\gamma = 0$. Furthermore, a non-zero subspace $\mathfrak{h} \subset \mathfrak{sl}_2(k)$ is a subalgebra if and only if it contains h, and all such are restricted ideals.*

Note that for $p \neq 2$, the height-one group scheme $\mathrm{SL}_2[F]$ is simple, while for $p=2$ it sits in a non-split extension $1 \to \mu_2 \to \mathrm{SL}_2[F] \to \alpha_2^{\oplus 2} \to 0$.

For any restricted Lie algebra $\mathfrak{h}$, the non-zero homomorphisms $\mathfrak{sl}_2(k) \to \mathfrak{h}$ correspond to non-zero triples (e,h,f) from $\mathfrak{h}$ satisfying the relations (2). Following Dynkin [18] and Bourbaki [10] we call them *$\mathfrak{sl}_2$-triples*. The triple is called *faithful* if the vectors $h,e,f \in \mathfrak{h}$ are linearly independent. Of course, the latter property is automatic in odd characteristics, by the preceding paragraph. Combing this with the Demazure–Gabriel Correspondence, we see that for a given scheme X the $\mathfrak{sl}_2$-triples in $H^0(X,\Theta_{X/k})$ correspond to the non-trivial actions of the height-one group scheme $\mathrm{SL}_2[F]$, and the faithful triples become the faithful actions.

Let us say that a scheme X *admits an $\mathfrak{sl}_2$-triple* if this holds for the restricted Lie algebra $\mathfrak{h} = H^0(X,\Theta_{X/k})$. In [44] we raised the question whether or not there are surfaces of general type admitting faithful $\mathfrak{sl}_2$-triples, and the goal of the present paper is to settle this problem.

Let us close this section with the following observation: Suppose X is a noetherian scheme with an $\mathfrak{sl}_2$-triple, with corresponding non-trivial action of $G = \mathrm{SL}_2[F]$. Recall that each non-zero vector $u = \alpha h + \beta e + \gamma f$ yields a subgroup scheme H of order p. The resulting scheme of fixed points X^H has the following regularity behaviour:

Lemma 1.4. *If the coordinates satisfy $\alpha^2 - \beta\gamma \neq 0$, the scheme X^H is regular at each point $x \in \mathrm{Reg}(X)$.*

Proof. Without loss of generality we may assume that X is the spectrum of a local noetherian ring that is regular. According to (3), the condition on the coordinates ensure that the group scheme H is of multiplicative type, and the assertion follows from [1], Proposition 5.1.16. □

This is particularly useful if X is regular and X^H has a divisorial part, which then forms a family of regular divisors depending in an algebraic way on the line $ku \subset \mathfrak{sl}_2(k)$. In one form or another, all what follows relies on this observation.

2. Pencils and $\mathfrak{sl}_2$-triples

Let k be a ground field of characteristic $p>0$, and X be a proper scheme, together with a surjective morphism $f: X \to \mathbb{P}^1$ endowed with a faithful $\mathfrak{sl}_2$-triple on X, and a compatible triple on $\mathbb{P}^1$. In other words, we have a commutative diagram

$$\begin{array}{ccc} \mathfrak{sl}_2(k) & \longrightarrow & H^0(X,\Theta_{X/k}) \\ \downarrow & & \downarrow \\ H^0(\mathbb{P}^1,\Theta_{\mathbb{P}^1/k}) & \longrightarrow & H^0(X, f^*(\Theta_{\mathbb{P}^1/k})) \end{array} \tag{4}$$

of restricted Lie algebras, where the upper horizontal map is injective, and the lower horizontal map is non-zero. In geometric terms, the height-one group scheme $G = \mathrm{SL}_2[F]$ acts faithfully on X, and also in a compatible way on the projective line, where the action is at least non-trivial. Note that if $\mathscr{O}_{\mathbb{P}^1} = f_*(\mathscr{O}_X)$, any G-action on X automatically induces an action on $\mathbb{P}^1$ by Blanchard's Lemma.

The image $\mathfrak{sl}_2(k)\cdot\mathscr{O}_X$ for the canonical map $\mathfrak{sl}_2(k)\otimes_k \mathscr{O}_X \to \Theta_{X/k}$ is a coherent subsheaf of the tangent sheaf that is stable under bracket and p-map, and one may call it *foliation*. Recall that an open set $U \subset \mathscr{O}_X$ containing the finite set $\mathrm{Ass}(\mathscr{O}_X)$ of all associated points is termed *schematically dense*. Throughout this section, we assume that our datum satisfies the following two conditions:

(i) The G-action on the projective line $\mathbb{P}^1$ is fixed point free.
(ii) The coherent sheaf $\mathfrak{sl}_2(k) \cdot \mathscr{O}_X$ is invertible on some schematically dense open set $U \subset X$.

Note that the first condition is vacuous in characteristic $p \geq 3$. The second condition holds if X is a RDP surfaces of general type, and more generally for geometrically normal surfaces satisfying $H^0(X, \omega_X^{\otimes -1}) = 0$, according to [44], Corollary 6.6. The combination of the above innocent assumptions has some remarkable consequences. Let us start with the following observation:

Lemma 2.1. *The composite map $\mathfrak{sl}_2(k) \cdot \mathscr{O}_X \subset \Theta_{X/k} \to f^*(\Theta_{\mathbb{P}^1/k})$ is bijective.*

Proof. The invertible sheaf $\Theta_{\mathbb{P}^1/k} = \mathscr{O}_{\mathbb{P}^1}(2)$ is globally generated by the image of the vertical map on the left in (4), because the G-action has no fixed points. It follows that that $\mathfrak{sl}_2(k) \cdot \mathscr{O}_X \to f^*(\Theta_{\mathbb{P}^1/k})$ is surjective. Seeking a contradiction, we assume that the kernel $\mathscr{N}$ is non-zero, and pick some $\zeta \in \mathrm{Ass}(\mathscr{N})$. Choose a surjection $\mathscr{O}_{X,\zeta}^{\oplus r} \to \Omega^1_{X/k,\zeta}$. The dual inclusion $\Theta_{X/k,\zeta} \subset \mathscr{O}_{X,\zeta}^{\oplus r}$ shows $\zeta \in \mathrm{Ass}(\mathscr{O}_X)$. On the other hand, since $\mathfrak{sl}_2(k) \cdot \mathscr{O}_X \to f^*(\Theta_{\mathbb{P}^1/k})$ is an isomorphism on some schematically dense open set, we must have $\zeta \notin \mathrm{Ass}(\mathscr{O}_X)$, contradiction. $\square$

In other words, the canonical map $\Theta_{X/k} \to f^*(\Theta_{\mathbb{P}^1/k})$ of coherent sheaves is split surjective, and the foliation $\mathfrak{sl}_2(k) \cdot \mathscr{O}_X$ defines a splitting. Dualizing the exact sequence $f^*(\Omega^1_{\mathbb{P}^1/k}) \to \Omega^1_{X/k} \to \Omega^1_{X/\mathbb{P}^1} \to 0$ thus induces a split extension coming with a decomposition

$$\Theta_{X/k} = \Theta_{X/\mathbb{P}^1} \oplus f^*(\Theta_{\mathbb{P}^1/k}), \tag{5}$$

where the second summand coincides with $\mathfrak{sl}_2(k) \cdot \mathscr{O}_X$ Informally speaking, the group scheme $\mathrm{Aut}_{X/k}$ looks infinitesimally like a product. A useful consequence:

Proposition 2.2. *The induced G-action on $\mathbb{P}^1$ is faithful, and the characteristic must be $p \geq 3$.*

Proof. In the commutative diagram (4), the upper vertical map is injective by our standing assumption. In light of the splitting (5), the composite mapping $\mathfrak{sl}_2(k) \to H^0(X, f^*(\Theta_{\mathbb{P}^1/k}))$ remains injective, and it follows that vertical map on the left is injective as well. This means that the G-action on the projective line is faithful. For $p = 2$, the three-dimensional Lie algebras $\mathfrak{sl}_2(k)$ and $\mathfrak{pgl}_2(k)$ are not isomorphic, which produces a contradiction. $\square$

Let $B = \mathrm{Spec}(\mathscr{O}_X)$ be the Stein factorization for $f : X \to \mathbb{P}^1$. By Blanchard's Lemma this carries another compatible $\mathfrak{sl}_2$-triple.

Proposition 2.3. *If X is geometrically normal and geometrically connected, then B is a twisted form of the projective line.*

Proof. It suffices to treat the case that k is algebraically closed. As X is normal and connected, the same holds for B. Since curves of genus $g \geq 2$, and elliptic curves alike, do not admit $\mathfrak{sl}_2$-triples, the only possibility is $B \simeq \mathbb{P}^1$. $\square$

The following will play a key role throughout:

Proposition 2.4. *If the proper scheme X is smooth then the morphism $f : X \to \mathbb{P}^1$ is smooth as well.*

Proof. Without loss of generality we may assume that the ground field k is algebraically closed. For each generic point $\eta \in X$, the corresponding irreducible component is a connected component $U \subset X$, and its image $f(U) \subset \mathbb{P}^1$ is G-stable. Since the action on the projective line is fixed point free, $f(\eta) \in \mathbb{P}^1$ must be the generic point, hence the morphism f is flat.

The sheaf of Kähler differentials $\Omega^1_{X/k}$ is locally free of rank $n = \dim(X)$. In the exact sequence

$$f^*(\Omega^1_{\mathbb{P}^1/k}) \xrightarrow{\varphi} \Omega^1_{X/k} \longrightarrow \Omega^1_{X/\mathbb{P}^1} \longrightarrow 0,$$

the terms on the left are locally free. The dual map $\varphi^\vee$ is split surjective, therefore $\varphi = \varphi^{\vee\vee}$ is split injective. It follows that $\Omega^1_{X/\mathbb{P}^1}$ is locally free of rank $n-1$, hence the fibers of f are smooth. Summing up, the morphism $f : X \to \mathbb{P}^1$ if flat with smooth fibers, hence smooth. □

Let us record the following consequence:

Corollary 2.5. *Suppose that the scheme X is smooth of dimension $m+1$, that the morphism $f : X \to \mathbb{P}^1$ is in Stein factorization, and that the dualizing sheaf ω_X restricts to a big invertible sheaf on the generic fiber X_η. Then X is a family of m-dimensional smooth varieties of general type parametrized by $\mathbb{P}^1$.*

Proof. It suffices to treat the case that the ground field k is algebraically closed. For each closed point $t \in \mathbb{P}^1$, the Adjunction Formula gives $\omega_{X_t} = \omega_X|X_t$. So by the Semicontinuity Theorem, the dualizing sheaf ω_{X_t} is big. The fiber X_t is smooth, according to the Theorem, and geometrically connected by $\mathscr{O}_{\mathbb{P}^1} = f_*(\mathscr{O}_X)$. Thus X is a family of smooth varieties of general type, obviously of dimension $m = \dim(X) - 1$. □

3. The inertia map

Let k be a ground field of characteristic $p > 0$, and Y be a reduced scheme of finite type endowed with an $\mathfrak{sl}_2$-triple, in other words, a non-trivial action of the height-one group scheme $G = \mathrm{SL}_2[F]$. We also write $\mathfrak{g} = \mathfrak{sl}_2(k)$ for the Lie algebra. The cartesian diagram

$$\begin{array}{ccc} I & \longrightarrow & Y \\ \downarrow & & \downarrow \Delta \\ G \times Y & \longrightarrow & Y \times Y, \end{array}$$

where the lower map is given by $(g, y) \mapsto (gy, y)$, defines the *inertia* I, which is a relative group scheme over Y whose structure morphism is finite, but usually far from flat. The fibers $I_y \subset G \otimes \kappa(y)$ can be seen are the *stabilizers* G_y with respect to y viewed as rational point on the base-change $Y \otimes \kappa(y)$, and correspond to the restricted subalgebra

$$\mathrm{Lie}(G_y) \subset \mathfrak{g} \otimes \kappa(y) = \mathfrak{sl}_2(\kappa(y)).$$

Let $Y_d \subset Y$, $0 \leq d \leq 3$ be the subscheme of points $y \in Y$ where the order the stabilizer takes the value $|I_y| = p^d$, defined in terms of suitable sheaves of Fitting ideals. This yields a stratification in the sense that $Y_{i+1} \subset \overline{Y}_i \smallsetminus Y_i$. The stratum

Y_0 is open, and can be seen as the locus where the G-action is free, whereas Y_3 is closed, and equals the *scheme of fixed points* Y^G.

In what follows, we assume that $Y_0 = Y_1 = \varnothing$*, and furthermore that* Y_2 *is dense.* Then our stratification simplifies to the closed scheme $Y^G = Y_3$ and the complementary open set $Y \smallsetminus Y^G = Y_2$. We see that $Y^G = \varnothing$ if and only if all the Lie algebras $\mathrm{Lie}(G_y)$, $y \in Y$ are two-dimensional. In this situation, the $\mathfrak{sl}_2$-triple is called *fixed point free.* In any case, the inertia $I \to Y$ becomes flat outside the closed set Y^G, and we obtain a family of two-dimensional restricted Lie algebras $\mathrm{Lie}(G_y)$, parametrized by the points $y \in Y \smallsetminus Y^G$ of the complementary open set. The resulting classifying map

$$f_{\mathrm{inert}} : Y \smallsetminus Y^G \longrightarrow \mathrm{Grass}^2(\mathfrak{g}), \quad y \longmapsto \mathrm{Lie}(G_y).$$

into the Grassmann variety of two-dimensional vector subspaces is of fundamental importance, and will be called the *inertia map.* Using the exterior power $\Lambda^2 \mathrm{Lie}(G_y)$, we can also see it as a map to $\mathrm{Grass}^1(\Lambda^2\mathfrak{g})$. Note that the Plücker embedding $\mathrm{Grass}^2(\mathfrak{g}) \subset \mathrm{Grass}^1(\Lambda^2\mathfrak{g})$ is an equality for dimension reasons, as both schemes are two-dimensional.

The inertia map is not constant, since our action is faithful. Neither can it be surjective, because there are subspaces that fail to be subalgebras. To determine the image we use the standard basis $h, e, f \in \mathfrak{g}$ for the following computation: Given linearly independent vectors $v = \alpha h + \beta e + \gamma f$ and $v' = \alpha' h + \beta' e + \gamma' f$, the 2-dimensional subspace $kv + kv'$ is already determined by the 2-vector $v \wedge v'$, and we obviously have

$$kv + kv' \subset \mathfrak{g} \text{ is a subalgebra} \quad \Longleftrightarrow \quad v \wedge v' \wedge [v, v'] = 0.$$

A straightforward computation in the Grassmann algebra using the relations (2) in our Lie algebra $\mathfrak{g} = \mathfrak{sl}_2(k)$ shows

$$v \wedge v' \wedge [v, v'] = (T_0^2 - 4T_1T_2) \cdot (h \wedge e \wedge f),$$

where $T_0 = \alpha\gamma' - \gamma\alpha'$ and $T_1 = \alpha\beta' - \beta\alpha'$ and $T_2 = \beta\gamma' - \gamma\beta'$ are the Plücker coordinates of the subspace. In our situation, the Plücker embedding $\mathrm{Grass}^2(\mathfrak{g}) \subset \mathrm{Grass}^1(\Lambda^2\mathfrak{g})$ is an equality, and the above computation reveals:

Proposition 3.1. *Inside* $\mathrm{Grass}^1(\Lambda^2\mathfrak{g})$*, the scheme of lines stemming from subalgebras* $\mathfrak{h} \subset \mathfrak{g}$ *is the curve defined by the quadratic equation* $T_0^2 - 4T_1T_2 = 0$.

Let us write $B = V_+(T_0^2 - 4T_1T_2)$ and call it the *curve of subalgebras.* Note that this is a copy of $\mathbb{P}^1$, which in characteristic two degenerates to a double line. In any case, the inertia map factors as

$$f_{\mathrm{inert}} : Y \smallsetminus Y^G \longrightarrow B = V_+(T_0^2 - 4T_1T_2) \subset \mathrm{Grass}^1(\Lambda^2\mathfrak{g}). \tag{6}$$

On the Grassmann variety $V = \mathrm{Grass}^2(\mathfrak{g})$, the quotient $\mathfrak{g} \otimes_k \mathscr{O}_V \to \mathscr{O}_V(1)$ by the tautological subsheaf is invertible, and the resulting $\mathfrak{g} \to H^0(V, \mathscr{O}_V(1))$ is bijective. Pulling this back to the open set $U = X \smallsetminus X^G$, we obtain an induced invertible quotient $\mathfrak{g} \otimes_k \mathscr{O}_U \to \mathscr{O}_U(1)$, and the G-action maps this to the tangent sheaf $\Theta_{U/k}$. Furthermore, the canonical map

$$\mathfrak{sl}_2(k) = \mathfrak{g} = H^0(V, \mathscr{O}_V(1)) \xrightarrow{f^*_{\mathrm{inert}}} H^0(U, \mathscr{O}_U(1)) \subset H^0(U, \Theta_{X/k}) \tag{7}$$

yields the restriction of the original $\mathfrak{sl}_2$-triple on Y. This has a surprising consequence:

Proposition 3.2. *In the above situation, the characteristic must be $p \geq 3$.*

Proof. Suppose $p = 2$. Then the curve of subalgebras becomes $V_+(T_1^2)$. Since our scheme Y is reduced, the inertia map factors over the line $L = V_+(T_1)$. It follows that the injection (7) factors over the two-dimensional vector space $H^0(L, \mathscr{O}_L(1))$, contradiction. □

So from this point onward we know $p \neq 2$. Our group scheme $G = \mathrm{SL}_2[F]$ acts on the scheme Y, on itself by conjugacy, and on the Lie algebra via the adjoint map $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$. Since G has height-one, the latter is already determined by its derivative

$$\mathrm{ad} : \mathfrak{g} \longrightarrow \mathfrak{gl}(\mathfrak{g}), \quad u \longmapsto (v \mapsto [u, v]).$$

To make this explicit, consider $v = \alpha h + \beta e + \gamma f$ and $v' = \alpha' h + \beta' e + \gamma' f$ as above. The adjoint action of $u = \pi h + \varphi e + \psi f$ on the plane $kv + kv'$ takes the form $v \wedge v' \mapsto [u, v] \wedge [u, v']$. A straightforward computation shows that this has the matrix interpretation

$$u \longmapsto \begin{pmatrix} -4\pi^2 & -2\pi\varphi & 2\pi\varphi \\ 4\pi\varphi & 2\varphi\psi & -2\varphi^2 \\ -4\pi\psi & -2\psi^2 & 2\varphi\psi \end{pmatrix} \tag{8}$$

with respect to the Plücker coordinates $T_0 = \beta\gamma' - \gamma\beta'$ and $T_1 = \alpha\gamma' - \gamma\alpha'$ and $T_2 = \alpha\beta' - \beta\alpha$. One easily checks directly that this G-action stabilizes the closed subscheme defined by $T_0^2 - 4T_1T_2 = 0$.

Proposition 3.3. *The inertia map $f_{\mathrm{inert}} : Y \smallsetminus Y^G \to \mathrm{Grass}^1(\Lambda^2\mathfrak{g})$ is G-equivariant, and the induced action on $B = V_+(T_0^2 - 4T_1T_2)$ is fixed point free.*

Proof. Equivariance of the inertia map can be checked for the functor of points, and then boils down to the set-theoretical fact $\sigma G_y \sigma^{-1} = G_{\sigma y}$. From (8) we see that $u = e$ has the effect $(1 : 2 : 1) \mapsto (2 : 0 : 0)$. Both of which belong to the curve of subalgebras, and it follows that the G-action is non-trivial. Since the group scheme G is simple, the action must be faithful. This identifies G with the Frobenius kernel of the automorphism group scheme for $V_+(T_0^2 - 4T_1T_2) = \mathbb{P}^1$, and the latter acts without fixed points. □

Properness of the curve of subalgebras has the following consequence:

Proposition 3.4. *No point $\zeta \in Y$ where the local ring $R = \mathscr{O}_{Y,\zeta}$ is regular and one-dimensional belongs to the scheme of fixed points Y^G.*

Proof. By our standing assumption, the open set $Y \smallsetminus Y^G$ is dense, so the inertia map is defined at the generic point $\eta \in Y$ corresponding to $F = \mathrm{Frac}(R)$. By the Valuative Criterion, the morphism $f_{\mathrm{inert}} : \mathrm{Spec}(F) \to \mathrm{Grass}^2(\mathfrak{g})$ extends to $\mathrm{Spec}(R)$. Hence the inertia map is defined on some open set U which contains both ζ and $U_0 = Y \smallsetminus Y^G$. The resulting morphism $U \to \mathbb{P}^1$ is equivariant, because this holds on a schematically dense open set U_0. All open sets are stable with respect to the infinitesimal group scheme G. Since the G-action on $V_+(T_0^2 - 4T_1T_2)$ is fixed point free, the same holds on U, and therefore $\zeta \notin Y^G$. □

So if Y is normal, the closed subscheme Y^G has codimension at least two. We now regard (6) as rational map $f_{\text{inert}} : Y \dashrightarrow B$ and seek to resolve the locus of indeterminacy. To this end consider the graph

$$\Gamma_{\text{inert}} \subset (Y \smallsetminus Y^G) \times B,$$

and write $\tilde{Y}$ for its schematic closure inside $Y \times B$. This closed subscheme is G-stable, according to [12], Lemma 2.1. By construction, it is an *equivariant compactification* of $Y \smallsetminus Y^G$, and the projection $r : \tilde{Y} \to Y$ is an *equivariant modification*. Note that $\tilde{Y}$ might be non-normal, even if Y is normal. The resulting inertia map

$$\tilde{f}_{\text{inert}} : \tilde{Y} \longrightarrow B = V_+(T_0^2 - 4T_1T_2) \subset \text{Grass}^2(\Lambda^2\mathfrak{g})$$

coincides with the composition $f_{\text{inert}} \circ r$. Our second main result is that this construction "resolves the fixed points" for the action of $G = \text{SL}_2[F]$:

Theorem 3.5. *The G-action on $\tilde{Y}$ is fixed point free. Moreover, the coherent subsheaf $\mathfrak{g} \cdot \mathscr{O}_{\tilde{Y}} \subset \Theta_{\tilde{Y}/k}$ is invertible, the canonical map $\Theta_{\tilde{Y}/k} \to \tilde{f}^*_{\text{inert}}(\Theta_{B/k})$ is surjective, and the foliation $\mathfrak{g} \cdot \mathscr{O}_{\tilde{Y}}$ yields a splitting.*

Proof. The first statement simply follows from the fact that the G-action on B is fixed point free. The remaining assertions are consequences of Lemma 2.1. ☐

In light of the above, we call $r : \tilde{Y} \to Y$ the *minimal resolution of fixed points*. It is quite remarkable that this exists.

4. Surfaces of general type

Let k be a ground field of characteristic $p > 0$. The following result answers a question raised in [44], and was our initial motivation for this research:

Theorem 4.1. *No smooth surface of general type has a faithful $\mathfrak{sl}_2$-triple.*

The proof requires some preparation, and will be given at the end of this section. Let us first consider proper normal surfaces Y satisfying merely $h^0(\mathscr{O}_Y) = 1$ and $h^0(\omega_Y^{\otimes -1}) = 0$. Obviously, the RDP surfaces of general type belong to this class. Throughout, we assume that Y has an $\mathfrak{sl}_2$-triple, in other words, a non-trivial action of the height-one group scheme $G = \text{SL}_2[F]$, with Lie algebra $\mathfrak{g} = \mathfrak{sl}_2(k)$. We first observe:

Lemma 4.2. *The scheme of fixed points Y^G is finite, and for all $y \in Y \smallsetminus Y^G$ the Lie algebra $\text{Lie}(G_y)$ is two-dimensional.*

Proof. Since the G-action is non-trivial, the corresponding foliation $\mathfrak{g} \cdot \mathscr{O}_Y \subset \Theta_{Y/k}$ is non-zero. The rank of this coherent sheaf is at least one, and bounded above by the so-called *foliation rank* $r \geq 0$, an invariant introduced in [44], Section 6. The condition $h^0(\omega_Y^{\otimes -1}) = 0$ ensures $r \leq 1$, according to loc. cit., Corollary 6.6. So for the function field $F = k(Y)$, the kernel of the canonical map $\mathfrak{g} \otimes_k F \to \mathfrak{g} \cdot F \subset \Theta_{Y/k,\eta}$ has dimension $d = 2$. By [44], Proposition 5.3 the Lie algebra of the generic stabilizer G_η has dimension $d = 2$. Our surface Y is normal, so by Proposition 3.4 the scheme of fixed points Y^G must be finite. ☐

Consequently, the theory developed in Section 3 applies in our situation. Let $\tilde{Y} \to Y$ be the minimal resolution of fixed points. Since the group scheme G is infinitesimal, the action on Y does not necessarily extend to the normalization, so $\tilde{Y}$ can be non-normal. However, the situation improves under suitable assumption:

Proposition 4.3. *Let $X \to \tilde{Y}$ be the normalization. If the surface Y has only rational singularities, the G-action extends to X, and the latter has only rational singularities.*

Proof. We recursively define a sequence

$$Y = Y_0 \longleftarrow Y_1 \longleftarrow Y_3 \longleftarrow \ldots$$

of proper normal G-surface Y_i and G-equivariant modifications as follows: To start with set $Y_0 = Y$. Suppose now that Y_i is already defined. If the G-action is fixed point free we set $Y_{i+1} = Y_i$. Otherwise pick some $z_i \in Y_i^G$, regard $Z_i = \{z_i\}$ as reduced closed subscheme, and define $Y_{i+1} = \mathrm{Bl}_{Z_i}(Y_i)$. Then the G-action on Y_i extends to Y_{i+1}, and the canonical morphism $Y_{i+1} \to Y_i$ is equivariant. According to [29], Proposition 8.1 the surface Y_{i+1} stays normal, and has only rational singularities. This concludes the recursive definition.

According to [29], Theorem 26.1 the rational map $f_{\mathrm{inert}} : Y \dashrightarrow \mathbb{P}^1$ becomes a morphism on Y_r for some index $r \geq 0$, so the G-action on Y_r is fixed point free. Therefore the canonical morphism $Y_r \to Y$ factors over $\tilde{Y}$, thus also over the normalization X. The induced morphism $h : X \to Y$ is in Stein factorization, so by Blanchard's Lemma there is a G-action on Y for which h is equivariant. It follows that the normalization map $X \to \tilde{Y}$ is equivariant.

To see that X has only rational singularities, choose a resolution of singularities $X' \to Y_r$. The Leray–Serre spectral sequence for the composition of $g : X' \to Y$ and $h : X \to Y$ gives an exact sequence

$$R^1(h \circ g)_*(\mathscr{O}_{X'}) \longrightarrow h_*(R^1 g_*(\mathscr{O}_{X'})) \longrightarrow R^2 h_*(\mathscr{O}_Y).$$

The term on the left vanishes because the singularities on X are rational, and the term on the right is trivial by the Theorem of Formal Functions. Consequently $R^2 g_*(\mathscr{O}_{X'}) = 0$. □

Note that the G-action on X is fixed-point free, since this already holds for $\tilde{Y}$. Applying Theorem 3.5 with X instead of Y, we see that $\mathfrak{g} \cdot \mathscr{O}_X \subset \Theta_{X/k}$ is invertible, the canonical map $\Theta_{X/k} \to \tilde{f}^*_{\mathrm{inert}}(\Theta_{B/k})$ is surjective, and the foliation $\mathfrak{g} \cdot \mathscr{O}_X$ yields a splitting.

Proposition 4.4. *Suppose Y is regular, or has only rational double points, or has only rational singularities. Then there is a G-modification $X \to Y$ with some normal surface X that has the respective properties, and where the G-action is fixed point free.*

Proof. Use the sequence $Y = Y_0 \leftarrow Y_1 \leftarrow \ldots \leftarrow Y_r$ of blowing-ups as in the preceding proof. The Y_i are regular, or have only rational double points, or have only rational singularities if the corresponding property holds for Y. We thus may set $X = Y_r$. □

Proof for Theorem 4.1. Seeking a contradiction, we assume that there is a smooth surface X of general type having a faithful $\mathfrak{sl}_2$-triple. Without loss of generality we may assume that k is algebraically closed. The characteristic must be $p \geq 3$, according to Proposition 3.2. By Proposition 4.4 me may further assume that the action of $G = \mathrm{SL}_2[F]$ is fixed point free. Consider the inertia map

$$f_{\mathrm{inert}} : X \longrightarrow B = V_+(T_0^2 - 4T_1T_2) \subset \mathrm{Grass}^1(\Lambda^2\mathfrak{g})$$

and its Stein factorization $f : X \to B'$. In light of Proposition 2.3 we have $B' = \mathbb{P}^1$. According to Corollary 2.5, X is a family of smooth curves of some genus $g \geq 2$, parameterized by the projective line. Let $\mathbb{P}^1 \to \mathscr{M}_g$ be the classifying map to the Deligne–Mumford stack. Such maps have zero-dimensional image: This was first established by Parshin [39] and Arakelov [2] in characteristic zero, and later extended to positive characteristics by Szpiro ([46], Theorem 3.3). It follows that our family of curves becomes constant on some étale covering of the projective line. Since k is separably closed, the only such covering is the identity. We thus have $X = C \times \mathbb{P}^1$ for some curve C. Let $D = \{c\} \times \mathbb{P}^1$ be some closed fiber with respect to the first projection. This curve is movable and ω_X is big, hence $(\omega_X \cdot D) > 0$. On the other hand, the Adjunction Formula gives $(\omega_X \cdot D) = -2$, contradiction. $\square$

5. Lefschetz pencils and alterations

Let k be a ground field of characteristic $p > 0$. The goal of this section is to construct RDP surfaces of general type admitting $\mathfrak{sl}_2$-triples. Note that by Blanchard's Lemma, this also produces $\mathfrak{sl}_2$-triples on their canonical model. In what follows, we work with the height-one group scheme $\mathrm{PGL}_2[F]$ instead of $\mathrm{SL}_2[F]$, in order to keep $p = 2$ included. Our main result in this direction is:

Theorem 5.1. *For each minimal surface S of general type and each integer $n \geq 0$, there is a purely inseparable alteration $X \to S$ with $\deg(X/S) = p^n$ by some RDP surface X of general type that has a faithful action of the group scheme $\mathrm{PGL}_2[F^n]$. Moreover, we may assume that all singularities are of type A_l with $l = p^n - 1$, and that the tangent sheaf $\Theta_{X/k}$ is locally free.*

The proof is constructive in nature and will be given in the course of this section, after some preparatory considerations.

Let $\mathscr{L}$ be a very ample invertible sheaf on our minimal surface S of general type, and $E \subset H^0(S, \mathscr{L})$ be some 2-dimensional vector subspace. Choose a basis $s_0, s_1 \in E$ and consider the resulting *pencil* of curves $C_t \subset S$ parameterized by $t \in \mathbb{P}^1$, and denote the *axis* by $Z = \bigcap C_t = C_0 \cap C_\infty$. It is called *Lefschetz pencil* if almost all C_t are smooth, each singular C_t has but one singularity that is furthermore étale locally given by the equation $xy = 0$, and the axis Z is finite and smooth. Upon blowing-up with the axis as center, the C_t become fibers of the *Lefschetz fibration* $\mathrm{Bl}_Z(S) \to \mathbb{P}^1$. According to [16], Exposé XVII, Theorem 2.5 the Lefschetz pencils form a dense open set inside the Grassmann variety of planes, at least after replacing $\mathscr{L}$ by $\mathscr{L}^{\otimes 2}$. Over infinite ground fields, such open sets contain rational points, so Lefschetz pencils indeed exist. This carries over to finite ground fields $k = \mathbb{F}_{p^\nu}$, by passing to higher tensor powers of $\mathscr{L}$, as explained in [40] and [38].

The blowing-up $\tilde{S} = \mathrm{Bl}_Z(S)$ is a smooth surface of general type, endowed with a Lefschetz fibration $\tilde{f} : \tilde{S} \to \mathbb{P}^1$.

Lemma 5.2. *The fibers of $\tilde{f} : \tilde{S} \to \mathbb{P}^1$ are stable curves of some genus $g \geq 2$.*

Proof. We may assume that k is algebraically closed. The assertion is obvious for the irreducible fibers, because $\omega_{\tilde{S}}$ is ample on each movable curve. Suppose now that $C_t = D \cup D'$ is reducible, and write $\tilde{D}, \tilde{D}' \subset \tilde{S}$ for the strict transforms of the irreducible components. From

$$0 = (\tilde{D} + \tilde{D}')^2 = \tilde{D}^2 + 2(\tilde{D} \cdot \tilde{D}') + \tilde{D}'^2 = \tilde{D}^2 + 2 + \tilde{D}'^2$$

we infer $\tilde{D}^2 = \tilde{D}'^2 = -1$, and thus $D^2, D'^2 \geq -1$. The Adjunction Formula gives $-2\chi(\mathscr{O}_D) = (\omega_S \cdot D) + D^2 \geq 0 - 1$, and thus $h^1(\mathscr{O}_D) \geq 1/2$, and likewise for D'. It follows that C_t is a stable curve of genus $g \geq 2$. ☐

We get a commutative diagram with cartesian squares:

$$\begin{array}{ccccc} X_n & \xrightarrow{h_n} & \tilde{S} & \longrightarrow & \bar{\mathscr{U}}_g \\ {\scriptstyle f_n}\downarrow & & \downarrow{\scriptstyle \tilde{f}} & & \downarrow \\ \mathbb{P}^1 & \xrightarrow[F^n]{} & \mathbb{P}^1 & \longrightarrow & \bar{\mathscr{M}}_g, \end{array}$$

where the vertical arrow on the right is the universal family of stable curves, the lower arrow on the right is the classifying map for the family $\tilde{S}$, and F^n denotes the relative Frobenius map for the projective line.

The induced projection $f_n : X_n \to \mathbb{P}^1$ is another family of stable curves of genus $g \geq 2$, and the projection $h_n : X_n \to \tilde{S}$ is finite universal homeomorphism that is flat of degree p^n. Also note that $\operatorname{Sing}(X_n/k)$ maps to the finite set $\operatorname{Sing}(\tilde{S}/\mathbb{P}^1)$. By Serre's Criterion, the surface X_n is normal.

Proposition 5.3. *Let $x \in X_n$ be a rational point mapping to $\operatorname{Sing}(\tilde{S}/\mathbb{P}^1)$. Then the local ring $\mathscr{O}_{X_n,x}$ is a rational double point of type A_l with $l = p^n - 1$. Moreover, X_n is a RDP surface of general type, the tangent sheaf $\Theta_{X_n/k}$ is locally free, and the evaluation pairing $\Theta_{X_n/k} \otimes \Omega^1_{X_n/k} \to \mathscr{O}_{X_n}$ is surjective.*

Proof. Let $\tilde{s} \in \tilde{S}$ and $t \in \mathbb{P}^1$ be the images of $x \in X_n$. The corresponding complete local ring $R = \mathscr{O}^\wedge_{\tilde{S},\tilde{s}}$ can be described by the equation $z = uv$, where $z \in \mathscr{O}^\wedge_{\mathbb{P}^1,z}$ is a uniformizer, and $u, v \in R$ is a system of regular parameters. The complete local ring at $x \in X$ then is described by $z^{p^n} = uv$, which yields a rational double point as claimed. From Proposition 6.3 below we obtain that the tangent sheaf $\Theta_{X_n/k}$ is locally free and the evaluation pairing $\Theta_{X_n/k} \otimes \Omega^1_{X_n/k} \to \mathscr{O}_{X_n}$ is surjective.

The relative dualizing sheaf for $F^n : \mathbb{P}^1 \to \mathbb{P}^1$ is invertible with $d = -2 + 2p^n \geq 0$, hence $\omega_{X_n} = h^*(\omega_{\tilde{S}}) \otimes f_n^*(\mathscr{O}_{\mathbb{P}^1}(d))$. The first tensor factor is big, the second effective, and it follows that the tensor product remains big. Using that the singularities of X_n are rational double points, we infer on the minimal resolution of singularities $X'_n \to X_n$, the dualizing sheaf stays big. Summing up, X_n is a RDP surface of general type. ☐

The group scheme $G_n = \operatorname{PGL}_2[F^n]$ acts on the projective line $\mathbb{P}^1$ in an canonical way, and the relative Frobenius map $F^n : \mathbb{P}^1 \to \mathbb{P}^1$ obviously factors over the quotient $\mathbb{P}^1/G_n$.

Lemma 5.4. *The induced map $\mathbb{P}^1/G_n \to \mathbb{P}^1$ is an isomorphism.*

Proof. By induction, it suffices to treat the case $n = 1$. The map in question is a map of degree one between proper regular curves, hence must be an isomorphism. □

Proof of Theorem 5.1. From the above observation, we see that in

$$X_n = \tilde{S} \times_{\mathbb{P}^1} (\mathbb{P}^1, F^n) = \mathrm{Bl}_Z(S) \times_{\mathbb{P}^1} (\mathbb{P}^1, F^n),$$

the G_n-action on the second factor induces an action on the fiber product, and the projection $f_n : X_n \to \mathbb{P}^1$ is equivariant. The action is faithful on the projective line, so the same holds on X_n. By construction, the canonical morphism $X_n \to S$ is a purely inseparable alteration of degree p^n, and we saw in Proposition 5.3 that X_n is a RDP surface of general type. □

Let us also make the following observation:

Proposition 5.5. *For each $x \in X_n$, the inertia group scheme G_x inside the base-change of $G = \mathrm{PGL}_2[F]$ has a two-dimensional Lie algebra.*

Proof. We may assume that x is a rational point, and it suffices to verify the corresponding statement for the image $b = f_n(x)$ on the projective line. Then the Lie algebra in question is the vector space $H^0(\mathbb{P}^1, \Theta_{\mathbb{P}^1/k}(-b))$, whose dimension is given by $h^0(\mathscr{O}_{\mathbb{P}^1}(1)) = 2$. □

In odd characteristics the canonical projection $\mathrm{SL}_2 \to \mathrm{PGL}_2$ is an isomorphism, and the above results can be summarized as follows:

Proposition 5.6. *For $p \neq 2$ and $n \geq 1$, the RDP surfaces X_n of general type carries a fixed point free $\mathfrak{sl}_2$-triple.*

6. Rational double points and tangent modules

We encountered RDP surfaces having a fixed point free $\mathfrak{sl}_2$-triple, where furthermore the tangent sheaf is locally free and the evaluation pairing with Kähler differentials is surjective. In light of the classification of rational double points, it is natural to ask which of them have one or both of these properties. Such questions showed up in many circumstances (for example [28], [49], [20], [42], [24], [43], [21], [26], [27], [31]), and the following observation shed more light on these issues.

Let us start with some homological algebra: Suppose R is a local noetherian ring with maximal ideal $\mathfrak{m}$ and residue field $k = R/\mathfrak{m}$. Recall that each finitely generated module M has *free resolutions*

$$\ldots \longrightarrow R^{\oplus r_3} \xrightarrow{\varphi_3} R^{\oplus r_2} \xrightarrow{\varphi_2} R^{\oplus r_1} \xrightarrow{\varphi_1} R^{\oplus r_0} \longrightarrow M \longrightarrow 0.$$

Such a resolution is called *minimal* if in φ_i viewed as a matrix the entries belong to the maximal ideal, or equivalently $\varphi_i(R^{\oplus r_i}) \subset \mathfrak{m}^{\oplus r_{i-1}}$. Then the ranks $r_i \geq 0$ depend only on M, and are called *Betti numbers* $b_i(M) = r_i$. Obviously, the *projective dimension* $\mathrm{pd}(M) \leq \infty$ is finite if and only if one of the Betti numbers vanishes. In what follows we write

$$M^* = \mathrm{Hom}(M, R) \quad \text{and} \quad E = \mathrm{Ext}^1(M, R),$$

and seek to understand when M^* is free, or the evaluation pairing $M^* \otimes M \to R$ given by $\psi \otimes a \mapsto \psi(a)$ is surjective, in dependence of Betti numbers of the involved modules.

Lemma 6.1. *Suppose $b_2(M)=0$. Then the following holds:*

(i) *The dual module M^* is free if and only if $b_3(E)=0$.*
(ii) *The evaluation paring $M^*\otimes M\to R$ is surjective if and only if $b_1(E)<b_0(M)$.*

Proof. Choose a minimal free resolution $0\to R^{\oplus s}\stackrel{{}^tJ}{\to}R^{\oplus r}\to M\to 0$. The resulting long exact sequence

$$0\longrightarrow M^*\longrightarrow R^{\oplus r}\stackrel{J}{\longrightarrow}R^{\oplus s}\longrightarrow E\longrightarrow 0$$

already gives (i). For the remaining statement, note that the entries in the matrices ${}^tJ$ and J belong to the maximal ideal, hence

$$b_0(M)=r\quad\text{and}\quad b_1(M)=s=b_0(E).$$

For assertion (ii), suppose first that $b_1(E)<r$. Without loss of generality, we may assume that all but the last standard basis vectors $e_1,\ldots,e_{r-1}\in R^{\oplus r}$ already generate the image of J. Then $R^{\oplus r-1}\times 0\stackrel{J}{\to}R^{\oplus s}\to E\to 0$ is the beginning of a free resolution. Write N for the syzygy module, and consider the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & N & \longrightarrow & R^{\oplus r-1} & \longrightarrow & \mathrm{Im}(J) & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \mathrm{id}} & & \\
0 & \longrightarrow & M^* & \longrightarrow & R^{\oplus r} & \longrightarrow & \mathrm{Im}(J) & \longrightarrow & 0.
\end{array}$$

The Snake Lemma provides a surjection $\psi:M^*\to R^{\oplus r}/R^{\oplus r-1}=R$ with kernel N. For the image $a\in M$ of the last basis vector $e_m\in R^{\oplus m}$ we then have $\psi(a)=1$.

Conversely, suppose there are $\psi\in M^*$, $a\in M$ such that $\psi(a)$ is a unit. Then our original module M splits off an invertible summand, and takes the form $M=M'\oplus R$. Clearly M' and M have the same projective dimension, and the Betti numbers are related by $b_0(M')=b_0(M)-1$ and $b_1(E)=b_1(E')=b_0(M')<b_0(M)$. □

There are further practical characterizations for freeness of M^* or surjectivity of $M^*\otimes M\to R$ in the case where R is a local complete intersection ring and

$$(9)\qquad b_2(M)=0\quad\text{and}\quad b_1(M)=1\quad\text{and}\quad \dim(E)=0\quad\text{and}\quad p>0.$$

Then the minimal resolution takes the from $0\to R\stackrel{{}^tJ}{\to}R^{\oplus r}\to M\to 0$. In the ensuing exact sequence $0\to M^*\to R^{\oplus r}\stackrel{J}{\to}R\to E\to 0$, the image of J is an $\mathfrak{m}$-primary ideal $\mathfrak{a}=(f_1,\ldots,f_r)$, and we write $\mathfrak{a}^{[p]}=(f_1^p,\ldots,f_r^p)$ for its *Frobenius power*. Now the three *modules of finite length* $R/\mathfrak{a}$ and $R/\mathfrak{a}^{[p]}$ and $\mathfrak{a}/\mathfrak{a}\mathfrak{m}$ become relevant:

Proposition 6.2. *Suppose that R is a complete intersection ring and the assumptions (9) hold. Then we have the following numerical characterizations:*

(i) *The dual module M^* is free if and only if* $\mathrm{length}(R/\mathfrak{a}^{[p]})=p^d\cdot\mathrm{length}(R/\mathfrak{a})$ *and* $\mathrm{depth}(R)=2$, *where* $d=\dim(R)$.
(ii) *The evaluation pairing $M^*\otimes M\to R$ is surjective if and only if the inequality* $\mathrm{length}(\mathfrak{a}/\mathfrak{a}\mathfrak{m})<r$ *holds.*

Moreover, if the local ring R is two-dimensional, the surjectivity of the pairing $M^*\otimes M\to R$ implies the freeness of the module M^*.

Proof. We start with (i). According to [36], Corollary 5.2.3 we have

$$\mathrm{pd}(E) < \infty \quad \Longleftrightarrow \quad \mathrm{length}(R/\mathfrak{a}) = p^d \cdot \mathrm{length}(R/\mathfrak{a}^{[p]}),$$

a fact already used in [42], Section 4. If the above equivalent conditions hold, the Auslander–Buchsbaum Formula ([19], Theorem 19.9) gives

$$\mathrm{pd}(M^*) = \mathrm{pd}(E) - 2 = \mathrm{depth}(R) - \mathrm{depth}(E) - 2 = \mathrm{depth}(R) - 2.$$

So if M^* is free, in other words $\mathrm{pd}(M^*) = 0$ and $\mathrm{pd}(E) = 2$, the former gives $\mathrm{depth}(R) = 2$, while the latter ensure $\mathrm{length}(R/\mathfrak{a}) = p^d \cdot \mathrm{length}(R/\mathfrak{a}^{[p]})$. The converse is likewise.

Suppose the evaluation pairing is surjective. By Lemma 6.1, the ideal $\mathfrak{a} = (f_1, \ldots, f_r)$ is already generated by less than r elements, and the Nakayama Lemma yields $\mathrm{length}(\mathfrak{a}/\mathfrak{m}\mathfrak{a}) < r$. The converse is likewise.

Finally, assume that $\dim(R) = \mathrm{depth}(R) = 2$, and that the equivalent conditions of (ii) hold. Then $r = 3$, and the ideal $\mathfrak{a}$ is already generated by $r - 1 = 2$ element. Write $\mathfrak{a} = (f, g)$. By assumption $R/\mathfrak{a}$ is zero-dimensional, hence the closed sets $V(f), V(g) \subset \mathrm{Spec}(R)$ have no common irreducible component. Using that R is Cohen–Macaulay, we infer that f, g form a regular sequence, so the Koszul complex

$$0 \longrightarrow \Lambda^2(F) \stackrel{{}^t(g,-f)}{\longrightarrow} \Lambda^1(F) \stackrel{(f,g)}{\longrightarrow} \Lambda^0(F) \longrightarrow R/\mathfrak{a} \longrightarrow 0$$

for $F = R^{\oplus 2}$ reveals $\mathrm{pd}(R/\mathfrak{a}) \leq 3$, hence M^* is free. □

The above observations from homological algebra apply in the following geometric situation: Fix a ground field k, for the moment of arbitrary characteristic $p \geq 0$, and consider local noetherian rings of the form

$$R = S^{-1}k[T_1, \ldots, T_m]/(P_1, \ldots, P_n),$$

where the multiplicative system S comprises all polynomials having non-zero constant terms, and $P_1, \ldots, P_n \notin S$. With $\mathfrak{a} = (P_1, \ldots, P_n)$ and $A = k[T_1, \ldots, T_m]$ we get an exact sequence

$$\mathfrak{a}/\mathfrak{a}^2 \longrightarrow \Omega^1_{A/k} \otimes_A R \longrightarrow \Omega^1_{R/k} \longrightarrow 0. \tag{10}$$

where $\mathfrak{a}/\mathfrak{a}^2$ is generated by the classes of P_i, and the map on the left is given by $P_i + \mathfrak{a}^2 \mapsto dP_i \otimes 1$.

If the $P_1, \ldots, P_m$ form a regular sequence in $S^{-1}k[T_1, \ldots, T_m]$, the module $\mathfrak{a}/\mathfrak{a}^2$ is free and the P_i provide a basis. If furthermore R is geometrically reduced, the map to $\Omega^1_{A/k} \otimes_A R$ injective, and the exact sequence (10) becomes a free resolution $0 \to R^{\oplus n} \to R^{\oplus m} \to \Omega^1_{R/k} \to 0$. It is minimal provided in addition that $\mathrm{edim}(R) = m$. Summing up, Lemma 6.1 and Proposition 6.2 apply for $M = \Omega^1_{R/k}$ under the above three assumptions, and give some information on the modules

$$\Theta_{R/k} = T^0_{R/k} = \mathrm{Hom}(\Omega^1_{R/k}, R) \quad \text{and} \quad T^1_{R/k} = \mathrm{Ext}^1(\Omega^1_{R/k}, R).$$

Note that the latter can be seen as the space of first order deformations for the scheme $U = \mathrm{Spec}(R)$, as explained in [4].

Suppose now that the ground field k is algebraically closed, and recall that the *rational double points* have been classified [5] and can be described by explicit equations $P(x, y, z) = 0$. The dual graph of the exceptional divisor on the minimal resolution of singularities corresponds to the *Dynkin diagrams* A_l $(l \geq 1)$ or D_l $(l \geq 4)$

or E_6, E_7, E_8. This already determines the formal isomorphism class, at least in characteristics $p \geq 7$. For the E-types at the remaining primes and the D-types in characteristic two one has to introduce *upper indices* to distinguish formal isomorphism class; these reflect the deformability of the singularity.

Proposition 6.3. *For the rational double points of type A_l, $l \geq 1$ we have:*

$$\Theta_{R/k} \text{ is free} \iff \Theta_{R/k} \otimes \Omega^1_{R/k} \to R \text{ is surjective} \iff l \equiv -1 \text{ modulo } p.$$

Moreover, in characteristic $p \geq 7$ there are no other rational double points where the tangent module is free or the paring is surjective, and for $p \geq 3$ there are at least no such of D-type.

Proof. In what follows, $P = P(x, y, z)$ is the polynomial defining the rational double point, $\mathfrak{b}$ the ideal in $A = k[x, y, z]$ generated by P and its partial derivative, and $\mathfrak{b}'$ is generated by P and the p-th powers of the partial derivatives.

For type A_l the four polynomials in question are

$$P = z^{l+1} - xy, \quad P_z = (l+1)z^l, \quad P_x = -y, \quad P_y = -x.$$

Suppose first $l + 1 \equiv 0$ modulo p. Then $\mathfrak{a} = \mathfrak{b}R = (x, y)$, and from Lemma 6.1 we see that the pairing $\Theta_{R/k} \otimes \Omega^1_{R/k} \to R$ is surjective, while Proposition 6.2 ensures that $\Theta_{R/k}$ is free. Suppose now that $l + 1 \not\equiv 0$. Then $\text{length}(A/\mathfrak{b}) = l$. On the other hand, $\bar{A} = k[x, y, z]/(P, x^p, y^p)$ is free as module over $k[x, y]/(x^p, y^p)$, and $1, z, \ldots, z^l$ provide a basis, giving $\text{length}(\bar{A}) = p^2 \cdot l$. In light of Proposition 6.2, it remains to check that $z^{lp} \in \bar{A}$ is non-zero. Indeed, otherwise $l + 1$ divides lp, which implies $l + 1 = p$, contradiction.

Suppose next that $p \neq 2$. For type D_l, $l \geq 4$ we have

$$P = z^2 + x^2y + xy^{l-1}, \quad P_z = 2z, \quad P_x = 2xy + y^{l-1}, \quad P_y = x^2 + (l-1)xy^{l-2}.$$

Inside the polynomial ring $A = k[x, y, z]$, we consider the ideals $\mathfrak{b} = (P, P_z, P_x, P_y)$ and $\mathfrak{b}' = \mathfrak{b}^{[p]} + (P)$. Our task is to show $\text{length}(A/\mathfrak{b}') \neq p^2 \cdot \text{length}(A/\mathfrak{b})$. First observe that $(l-1)xP_x - yP_y = 2(l-1)(1-y)x^2$. It follows that (P_z, P_x, P_y) is also generated by

$$Q_1 = z \quad \text{and} \quad Q_2 = x^2, \quad \text{and} \quad Q_3 = 2xy + y^{l-1},$$

and thus $\text{length}(A/\mathfrak{b}) = 2(l-1)$. Likewise we see that $\bar{A} = A/(P, Q_2^p, Q_3^p)$ has length $p^2 \cdot 2(l-1)$. It thus suffices to verify that the class of $Q_1^p = z^p$ in $\bar{A}$ does not vanish. The latter is freely generated, as module over $k[x]/(x^{2p})$, by the monomials y^iz^j with $0 \leq i < (l-1)p$ and $0 \leq j < 2$. Computing modulo P, we see

$$z^p = (z^2)^{(p-1)/2} \equiv (-x^2y - xy^{l-1})^{(p-1)/2} = \pm x^{(p-1)/2}y^{(l-1)(p-1)/2} + (\ldots),$$

where the remaining summand have lower degree in y. It follows that $z^p \in \bar{A}$ is indeed non-zero.

It remains to treat type E_6, E_7, E_8 in characteristic $p \geq 7$, where the respective polynomials are $P = z^2 + x^3 + y^4$ and $P = z^2 + x^3 + xy^3$ and $P = z^2 + x^3 + y^5$. The arguments are as above, and left to the reader. □

As usual, the small primes are in need of special attention. There are only finitely many rational double points of E-type; using computer algebra [9], we compute the lengths for the relevant modules $R/\mathfrak{a}, R/\mathfrak{a}^{[p]}$ together with $\mathfrak{a}/\mathfrak{am}$ occurring in

Proposition 6.2. The results are collected in table 2, and immediately yield the following:

Proposition 6.4. *The rational double points of E-type where the tangent module $\Theta_{R/k}$ is free are precisely for*

$$E_8^0\ (p=5) \quad \textit{and} \quad E_6^0, E_7^0, E_8^0\ (p=3) \quad \textit{and} \quad E_6^0, E_7^0, E_7^1, E_7^2, E_8^0, E_8^1, E_8^2\ (p=2).$$

The evaluation pairing $\Theta_{R/k} \otimes \Omega^1_{R/k} \to R$ is surjective precisely for E_8^0 $(p=5)$ and E_6^0, E_7^0, E_8^0 $(p \leq 3)$.

The hardest challenge is to understand the rational double points of D-type in characteristic two. Using computer algebra to compute examples, one immediately comes up with the following:

Proposition 6.5. *In characteristic $p=2$, the rational double points of type D_l^r, $l \geq 4$ have the following behavior with regards to the tangent module $\Theta_{R/k}$ and the pairing $\Theta_{R/k} \otimes \Omega^1_{R/k} \to R$:*

(i) *For $r=0$, the tangent module is free and the pairing is surjective.*
(ii) *For $l=2n$ even and $1 \leq r \leq n-1$, the tangent module is free but the pairing is not surjective.*
(iii) *For $l=2n+1$ odd and $1 \leq r \leq n-1$, neither the tangent module is free nor the pairing is surjective.*

Proof. For D_{2n}^0 and D_{2n+1}^0 the respective polynomials are $P=z^2+x^2y+xy^n$ and $P=z^2+x^2y+y^nz$. One of the partial derivatives P_z or P_x vanishes, so (i) holds by Lemma 6.1.

Suppose now $r \geq 1$ and $l=2n$. Then the partial derivatives of the defining polynomial $P=z^2+x^2y+xy^n+xy^{n-r}z$ admit factorizations

$$P_z = x \cdot y^{n-r} \quad \text{and} \quad P_x = y^{n-r} \cdot (y^r+z) \quad \text{and} \quad P_y = x \cdot Q$$

with $Q = x+ny^{n-1}+(n-r)y^{n-r-1}z$, and thus can be seen as maximal minors of the 3×2-matrix

$$\varphi_1 = {}^t\begin{pmatrix} y^r+z & x & 0 \\ Q & 0 & y^{n-r} \end{pmatrix}.$$

Extending the matrix with $\varphi_0 = (P_z, P_x, P_y)$ as first row and applying Laplace Expansion, we get a complex $0 \to R^{\oplus 2} \stackrel{\varphi_1}{\to} R^{\oplus 3} \stackrel{\varphi_0}{\to} \mathfrak{a} \to 0$ for the ideal $\mathfrak{a} \subset R$ generated by the partial derivatives. According to the Hilbert–Burch Theorem ([19], Theorem 20.15), this complex is actually exact, hence $\Theta_{R/k}$ is free. All entries of φ_1 belong to the maximal ideal, so Lemma 6.1 tell us that $\Theta_{R/k} \otimes \Omega^1_{R/k} \to R$ is not surjective. This settles (ii).

It remains to treat the case $r \geq 1$ and $l=2n+1$. Now the defining polynomial is $P=z^2+x^2y+y^nz+xy^{n-r}z$, with partial derivatives

$$P_z = y^n + xy^{n-r}, \quad P_x = y^{n-r}z, \quad P_y = x^2+ny^{n-1}z+(n-r)xy^{n-r-1}z.$$

To simplify notation set $\nu = n - r$, and consider the following three derivations

$$
\begin{aligned}
\delta_1 &= z\frac{\partial}{\partial z} + (x + y^r)\frac{\partial}{\partial x}, \\
\delta_2 &= (\nu+1)y^{n-r}z\frac{\partial}{\partial z} + (\nu y^n + z)\frac{\partial}{\partial x} + y^{n-r+1}\frac{\partial}{\partial y}, \qquad (11)\\
\delta_3 &= (xz + y^r z + \nu y^{n-r-1}z)\frac{\partial}{\partial z} + (y^{2r} + (\nu+1)y^{n-1}z)\frac{\partial}{\partial x} + y^{n-r}z\frac{\partial}{\partial y}
\end{aligned}
$$

of $A = k[x,y,z]$. One easily checks that each of them stabilizes the principal ideal generated by P, and thus give rise to elements $\bar{\delta}_1, \bar{\delta}_2, \bar{\delta}_3 \in \Theta_{R/k}$. Let $J_2 \in \mathrm{Mat}_3(A)$ be the coefficient matrix for the above derivations, and $J_1 = (P_z, P_x, P_z)$ be the Jacobi matrix. Then

$$
J_1 \cdot J_2 \equiv 0 \mod P, \qquad J_2 \equiv \begin{pmatrix} z & 0 & 0 \\ x & z & 0 \\ 0 & 0 & 0 \end{pmatrix} \mod \mathfrak{m}_A.
$$

Seeking a contradiction, we now assume that $\Theta_{R/k}$ is free. Suppose for the moment that $b_1(T^1_{R/k}) = 3$. Then $R^{\oplus 3} \stackrel{J_1}{\to} R \to T^1_{R/k} \to 0$ is the beginning of a minimal resolution, and for all $\bar{\delta}$ from $\Theta_{R/k} \subset R^{\oplus 3}$ the coordinates belong to $\mathfrak{m}_R$. Using that the first two rows in J_2 are linearly independent modulo $\mathfrak{m}_A^2$, we infer that $\bar{\delta}_1, \bar{\delta}_2 \in \Theta_{R/k}$ form a basis. We thus have $\bar{\delta}_3 = Q_1\bar{\delta}_1 + Q_2\bar{\delta}_2$ for some polynomials $Q_i = Q_i(x,y,z)$. Comparing coefficients in the third coordinate, we arrive at $y^{n-r}z \in (y^{n-r+1}, P)$. On the other hand, the residue class ring $k[x,y,z]/(y^{n-r+1}, P)$ is free as $k[x]$-module, with basis given by monomials $y^i z^j$ with $0 \leq i < n - r + 1$ and $0 \leq j < 2$. So $y^{n-r}z$ is a basis member, contradiction.

It remains to verify $b_1(T^1_{R/k}) = 3$. Seeking a contradiction, we assume that the ideal $\mathfrak{b} = (P, P_z, P_x, P_y)$ in $A = k[x,y,z]$ is generated by P together with only two partial derivatives. To use Gröbner bases techniques, introduced weights

$$
\deg(x) = \deg(z) = n - 1 \quad \text{and} \quad \deg(y) = 1,
$$

and define a *monomial ordering* $x^i y^r z^j \succ x^{i'} y^{r'} z^{j'}$ if $\deg(x^i y^r z^j) < \deg(x^{i'} y^{r'} z^{j'})$, and in case of equality use the lexicographic ordering with $x \succ z \succ y$. By design, the leading terms are as follows:

$$
P = z^2 + (\ldots), \quad P_z = y^n + (\ldots), \quad P_x = y^{n-r}z, \quad P_y = x^2 + (\ldots). \qquad (12)
$$

We claim that the above generators form a *standard basis*. Recall that this notion was introduced, in one form or another, by Hironaka [25], Buchberger [13] and Grauert [22], and is now usually seen as a variant of *Gröbner bases* in the setting of localized or completed polynomial rings. See for example [23] or [14] for more details. To apply the *Buchberger Criterion* we have to compute S-polynomials and check that division with reminder reduces them to zero: This indeed holds for

$$
S(P_z, P_x) = xP_x \quad \text{and} \quad S(P, P_x) = (x^2 y + y^n + xy^{n-r})P_x.
$$

The other S-polynomials also reduce to zero because the leading terms of the generators are pairwise prime, ([15], Section 2, §9, Proposition 4). Thus (12) from a standard basis, hence

$$
\mathrm{length}(A/\mathfrak{b}) = \mathrm{length}(k[x,y,z]/(z^2, y^n, y^{n-r}z, x^2)) = 4n - 2r,
$$

	E_6^0	E_6^1	E_7^0	E_7^1	E_7^2	E_7^3	E_8^0	E_8^1	E_8^2	E_8^3	E_8^4
$p=2$	$8,32$ 2	$6,28$ 3	$14,56$ 2	$12,48$ 3	$10,40$ 3	$8,35$ 3	$16,64$ 2	$14,56$ 3	$12,48$ 3	$10,44$ 3	$8,37$ 3
$p=3$	$9,81$ 2	$7,71$ 3	$9,81$ 2	$7,75$ 3			$12,108$ 2	$10,99$ 3	$8,85$ 3		
$p=5$	$6,173$ 3		$7,198$ 3				$10,250$ 2	$8,239$ 3			

TABLE 2. Lenghts of $R/\mathfrak{a}, R/\mathfrak{a}^{[p]}$ and $\mathfrak{a}/\mathfrak{am}$ for RDP of E-type for $p \leq 5$.

as stated without proof in [5]. Writing $\mathfrak{b}_z = (P, P_x, P_y)$ and $\mathfrak{b}_x = (P, P_z, P_y)$ and $\mathfrak{b}_y = (P, P_z, P_x)$, we likewise see that the generators form a standard basis, and obtain

$$\text{length}(A/\mathfrak{b}_z) = \text{length}(A/\mathfrak{b}_y) = \infty \quad \text{and} \quad \text{length}(A/\mathfrak{b}_x) = 4n.$$

In turn, one cannot remove in $\mathfrak{b} = (P, P_z, P_x, P_y)$ a partial derivative to obtain a smaller generating set, and thus $b_1(T^1_{R/k}) = 3$. □

Remark 6.6. Note that the statements pertaining to the freeness of $\Theta_{R/k}$ already appears in the work of Graf [21], with an argument relying on a computer program. This is fine in each individual case, but we could not see how this settles infinite families like D_n. Also note that Matsumoto ([33], Proposition 4.7 and [34], Theorem 3.3) obtained similar results on the pairing $\Theta_{R/k} \otimes \Omega^1_{R/k} \to R$. The difference is that he assumed that surjectivity is obtained with p-closed rather than an arbitrary derivations and he did not relate this with the freeness of the tangent sheaf. Our arguments are more general and they can be applied to treat other classes of singularities too.

7. Rational double points with $\mathfrak{sl}_2$-triples

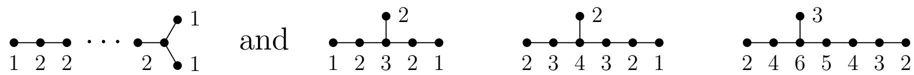
Let k be an algebraically closed ground field of characteristic $p > 0$. In this final section we take up the natural question which rational double points admit $\mathfrak{sl}_2$-triples. This is of local nature, and we work in the following setting: Let Y be a connected normal surface that is separated and of finite type, endowed with an $\mathfrak{sl}_2$-triple, in other words, a non-trivial action of the height-one group scheme $G = \text{SL}_2[F]$. Suppose $y \in Y$ is a RDP singularity and that there are no further singularities, and write $f : X \to Y$ for the minimal resolution of singularities. In characteristic two the following arguments are somewhat inconclusive, and for the sake of exposition we assume $p \neq 2$ throughout.

Proposition 7.1. *The rational double point $y \in Y$ is of type A_1, or belongs to the following list:*

$$A_l \ (l \equiv -1) \quad \textit{and} \quad E_8^0 \ (p = 5) \quad \textit{and} \quad E_6^0, E_6^1, E_7^0, E_8^0, E_8^1 \ (p = 3).$$

Here the above congruence is modulo p. Moreover, the G-action on Y does not extend to X, except for type A_1.

Proof. Let $E_1, \ldots, E_l \subset X$ be the exceptional divisors. According to [3], Theorem 4 the schematic fiber $Z = f^{-1}(y)$ coincides with the fundamental cycle of the singularity. For type A_l the cycle is reduced, while for D_l and E_6, E_7, E_8 the multiplicities in $Z = \sum_{i=1}^{l} m_i E_i$ are as follows:



For $l = 1$ our singularity is of type A_1, and there is nothing to prove. Suppose now $l \geq 2$. Since $p \neq 2$, we find by inspection in each case a pair of indices $r \neq s$ such that $E_r \cap E_s$ is non-empty and $p \nmid m_r m_s$.

Seeking a contradiction, we suppose that the G-action on Y lifts to X. Then the fiber $Z = f^{-1}(y)$ is G-stable ([30], Corollary 2.18). In turn, the $m_i E_i$ are G-stable ([12], Lemma 2.3), and the same holds for $D = m_r E_r \cup m_s E_s$. It then follows that $D_{\text{red}} = E_r \cup E_s$ is G-stable ([48], Proposition 4.2), therefore the unique point $x \in E_r \cap E_s$ is G-fixed. If the G-action on $E_r = \mathbb{P}^1$ is non-trivial, the induced map $G \to \mathrm{PGL}_2[F]$ is an isomorphism, and we see that x is not fixed, contradiction. So E_r is G-fixed, and the same holds for E_s. In particular, the fixed scheme X^G contains the singular curve $E_r \cup E_s$. On the other hand, for the multiplicative group schemes $\mu_p \subset G$ the fixed scheme X^{μ_p} is smooth ([1], Proposition 5.1.16), contradiction.

This establishes that the G-action on Y does not lift to the minimal resolution of singularities. Consequently, the canonical map $f_*(\Theta_{X/k}) \to \Theta_{Y/k}$ is not bijective. According to Hirokado's result ([24], Theorem 1.1), this holds precisely for the asserted types. □

The rings $A = k[u, v_1, \ldots, v_n]/\mathfrak{a}$ where the ideal is generated by polynomials of the form $P = u^n - f(v_1, \ldots, v_n)$ with $n \equiv 0$ admit an obvious $\mathfrak{sl}_2$-triple, given by the derivations $h = 2u\partial/\partial u$ and $e = u^2\partial/\partial u$ and $f = \partial/\partial u$, a fact already used in Section 5. In light of Artin's classification [5] of the rational double points in terms of equations $P(x, y, z) = 0$, we immediately see that the types

$$A_l \ (l \equiv -1) \quad \text{and} \quad E_8^0 \ (p = 5) \quad \text{and} \quad E_6^0, E_7^0, E_8^0 \ (p = 3) \tag{13}$$

admit $\mathfrak{sl}_2$-triples. For A_1 the equation is $z^2 - xy = 0$, and then

$$h = -2x\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial y} \quad \text{and} \quad e = 2z\frac{\partial}{\partial x} + y\frac{\partial}{\partial z} \quad \text{and} \quad f = 2z\frac{\partial}{\partial y} + x\frac{\partial}{\partial z}$$

yields an $\mathfrak{sl}_2$-triple. The cases E_6^1, E_8^1 $(p = 3)$ remain unclear to us.

Write $R = \mathscr{O}_{Y,y}$ for the local ring of the rational double point $y \in Y$. If the G-action is fixed-point free, the evaluation pairing $\Theta_{R/k} \otimes \Omega^1_{R/k} \to R$ is surjective. Conversely, suppose that the evaluation pairing is surjective. According to Proposition 6.3 and Proposition 6.4, this happens precisely for the types listed in (13). A priori, $y \in Y$ might be G-fixed. Strangely, this is only possible in two cases:

Corollary 7.2. *It the evaluation pairing $\Theta_{R/k} \otimes \Omega^1_{R/k} \to R$ is surjective and the singularity $y \in Y$ is G-fixed, the type is E_6^0, E_8^0 $(p = 3)$.*

Proof. The G-action lifts to the blowing-up $Y' = \mathrm{Bl}_y(Y)$. This is a normal surface whose singularities are rational double points. It turns out that there is at most one singularity $y' \in Y'$, and the types are related as follows:

Y	A_l	E_6	E_7	E_8
Y'	A_{l-2}	A_5	D_6	E_7

The minimal resolution $X \to Y$ factors over Y', and by our Proposition, the G-action does not lift to X. Applying Hirokado's result ([24], Theorem 1.1) to the morphism $X \to Y'$, we see that we must have $p = 3$ and $y' \in Y'$ has type A_5 or E_7^0. Our assertion follows. □

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